University of Ottawa
MAT 1332C Midterm Exam

Feb. 11, 2009. Duration: 80 Minutes. Instructor: Frithjof Lutscher

Family Name:______________________

First Name:_______________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved TI-30 calculator is allowed.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Where it is possible to check your work, do so.
- Good Luck!

Student number:____________________, Total marks: ___________ out of 27 (+2 bonus)

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [3 points] Consider the two functions

$$f(x) = \frac{x}{1+x}, \qquad g(x) = \frac{1}{2}x.$$

1. (1 point) Show that the functions intersect at the two points $x_1 = 0$ and $x_2 = 1$.
2. (2 points) Find the area enclosed by the two functions between the two points of intersection.

Solution

$$f(x) = g(x) \quad \Leftrightarrow \quad \frac{x}{1+x} = \frac{1}{2}x \quad \Leftrightarrow \quad x^2 - x = 0.$$

The solutions of the quadratic equation are $x_1 = 0$, and $x_2 = 2$. Hence, the functions intersect only at these two points.

The slope of f at zero is $f'(0) = 1$, which is bigger than the slope of g at zero. Hence, the function f is above the function g on the interval $(0, 1)$. Then the area between the two graphs is given by

$$\int_0^1 |f(x) - g(x)|dx = \int_0^1 [f(x) - g(x)]dx = \int_0^1 \left[1 - \frac{1}{1+x} - \frac{1}{2}x\right] dx =$$

$$\left[x - \ln|1+x| - \frac{1}{4}x^2\right] |_0^1 = 1 - \ln(2) - 1/4 = 3/4 - \ln(2) \approx 0.0569.$$

Solution version B

$$\int_0^1 |f(x) - g(x)|dx = \int_0^1 \left[1 - \frac{2}{2+x} - \frac{1}{3}x\right] dx = 5/6 - 2\ln(3/2) \approx 0.224.$$

Solution version C

$$\int_0^1 |f(x) - g(x)|dx = \int_0^1 \left[1 - \frac{3}{3+x} - \frac{1}{4}x\right] dx = 7/8 - 3\ln(4/3) \approx 0.0120.$$

Question 2. [5 points] Suppose that a tree grows in height according to the equation

$$\frac{dH}{dt} = 2e^{-0.1t}, \qquad H(0) = 7.$$

The units are in metres.

1. (2 points) How much does the tree grow between $t = 0$ and $t = 4$?
2. (2 points) Is the growth until $t = \infty$ finite or infinite?
3. (1 point) Will the total height of the tree ever reach 30 m?

Solution
Growth between $t = 0$ and $t = 4$:

$$\int_0^4 \frac{dH}{dt} dt = 2\int_0^4 e^{-0.1t} dt = \frac{2}{-0.1} e^{-0.1t}|_0^4 = 20(1 - e^{-0.4}) \approx 6.59.$$

Growth until $t = \infty$:

$$\int_0^\infty \frac{dH}{dt} dt = \lim_{T\to\infty} 2\int_0^T e^{-0.1t} dt = \lim_{T\to\infty} -20e^{-0.1t}|_0^T = \lim_{T\to\infty} 20(1 - e^{-0.1T}) = 20.$$

Since the tree starts at a height of 7 m and grows a maximum of 20 m, it will not reach 30 m.

Solution version B

$$\int_0^4 \frac{dH}{dt} dt = 4\int_0^4 e^{-0.2t} dt = 20(1 - e^{-0.8}) \approx 11.01.$$

Growth until $t = \infty$ is the same as above, as is the last part.

Solution version C

$$\int_0^4 \frac{dH}{dt} dt = 5\int_0^4 e^{-0.25t} dt = 20(1 - e^{-1}) \approx 12.64.$$

Growth until $t = \infty$ is the same as above, as is the last part.

Question 3. [4 points] Find the indefinite integral

$$\int \frac{9x+29}{x^2+2x-15}dx.$$

Solution

The denominator factors as $x^2+2x-15=(x-3)(x+5)$, hence we have two distinct roots. Then we use partial fractions

$$\frac{9x+29}{x^2+2x-15}=\frac{A}{x-3}+\frac{B}{x+5}=\frac{(A+B)x+5A-3B}{x^2+2x-15},$$

and compare coefficients to get $A=7, B=2$. Then we can integrate

$$\int \frac{9x+29}{x^2+2x-15}dx=\int \left[\frac{7}{x-3}+\frac{2}{x+5}\right]dx=7\ln|x-3|+2\ln|x+5|+C.$$

Solution version B

The denominator factors as $x^2-3x-10=(x+2)(x-5)$, hence we have two distinct roots. Then we use partial fractions

$$\frac{11x+1}{x^2-3x-10}=\frac{A}{x+2}+\frac{B}{x-5}=\frac{(A+B)x-5A+2B}{x^2-3x-10},$$

and compare coefficients to get $A=3, B=8$. Then we can integrate

$$\int \frac{11x+1}{x^2-3x-10}dx=\int \left[\frac{3}{x+2}+\frac{8}{x-5}\right]dx=3\ln|x+2|+8\ln|x-5|+C.$$

Solution version C

The denominator factors as $x^2-x-12=(x+3)(x-4)$, hence we have two distinct roots. Then we use partial fractions

$$\frac{11x-2}{x^2-x-12}=\frac{A}{x+3}+\frac{B}{x-4}=\frac{(A+B)x-4A+3B}{x^2-x-12},$$

and compare coefficients to get $A=5, B=6$. Then we can integrate

$$\int \frac{11x-2}{x^2-x-12}dx=\int \left[\frac{5}{x+3}+\frac{6}{x-4}\right]dx=5\ln|x+3|+6\ln|x-4|+C.$$

Question 4. [2 points] Does the following improper integral converge or diverge? If it converges, give its value.

$$\int_1^2 \frac{3}{\sqrt[3]{x-1}}dx.$$

Solution

The denominator is zero for $x = 1$. We substitute and use the definition for improper integrals to get

$$\int_1^2 \frac{3}{\sqrt[3]{x-1}}dx = \int_0^1 3u^{-1/3}du = \lim_{a\to 0^+}\int_a^1 3u^{-1/3}du = \lim_{a\to 0^+}\frac{9}{2}u^{2/3}|_a^1 = \lim_{a\to 0}\frac{9}{2}(1-a^{2/3}) = \frac{9}{2}.$$

Hence, the integral converges and its value is $9/2$.

Solution version B

The denominator is zero for $x = 2$. We substitute and use the definition for improper integrals to get

$$\int_2^3 \frac{5}{\sqrt[3]{x-2}}dx = \int_0^1 5u^{-1/3}du = \lim_{a\to 0^+}\int_a^1 5u^{-1/3}du = \lim_{a\to 0^+}\frac{15}{2}u^{2/3}|_a^1 = \lim_{a\to 0}\frac{15}{2}(1-a^{2/3}) = \frac{15}{2}.$$

Hence, the integral converges and its value is $15/2$.

Solution version C

The denominator is zero for $x = 1$. We substitute and use the definition for improper integrals to get

$$\int_1^3 \frac{7}{\sqrt[3]{x-1}}dx = \int_0^2 7u^{-1/3}du = \lim_{a\to 0^+}\int_a^2 7u^{-1/3}du = \lim_{a\to 0^+}\frac{21}{2}u^{2/3}|_a^2 = \lim_{a\to 0}\frac{21}{2}(2^{2/3}-a^{2/3}) = \frac{21}{2}\sqrt[3]{4}.$$

Hence, the integral converges and its value is $21/2\sqrt[3]{4} \approx 16.67$.

Question 5. [4 points + 2 bonus points] An intravenous drip is the continuous infusion of fluids into the human blood stream, which is often performed in hospitals to correct electrolyte balance. We denote by M the mass (in milligrams) of electrolytes that enter the bloodstream per unit time. These electrolytes are taken up by the body at a rate k. We denote the concentration of electrolytes in the blood stream by E (in milligrams per litre). The the equation for E is

$$\frac{dE}{dt} = aM - kE,$$

where a converts mass of electrolytes into concentration in the blood stream, i.e. $1/a$ is the amount of blood of the person. Choose the parameter values: $a = 5, k = 1/5, M = 10$.

1. (2 points) Find the steady state E^*. Determine its stability, using the derivative test. What is the concentration in the blood stream in the long run?
2. (2 points) Draw the phase line diagram and sketch the solution curve $E(t)$, starting at $E(0) = 50$.
3. (**Bonus 2 points**) Find an explicit solution to the equation with initial value $E(0) = 50$. [Hint: from the phase line diagram, you know that the solution satisfies $E(t) < 250$ for all $t > 0$.]

Solution

We denote the right hand side of the differential equation by $f(E) = aM - kE$. The steady state is given by the equation $f(E^*) = 0$. Hence

$$aM - kE^* = 0 \quad \Leftrightarrow \quad E^* = \frac{aM}{k} = 250.$$

To check stability, we have to evaluate the derivative of f at the steady state.

$$f'(E) = -k = -0.2 < 0$$

Hence, the derivative is negative (independent of the steady state). Then the steady state is stable. In the long run, the concentration will approach the steady state value of $aM/K = 250$ mg/l. The phase-line diagram and the solution curve are in Figure 1.

For the bonus question, we separate variables and integrate

$$\int \frac{dE}{aM - kE} = \int dt \quad \Longrightarrow \quad \frac{-1}{k} \ln|aM - kE| = t + C.$$

We take exponentials on both sides to get

$$|aM - kE| = De^{-kt}, \qquad D = e^{-kC}.$$

Since the solution starts between 0 and aM/k, it will remain between 0 and aM/k, so that we can drop the absolute values. Then we solve for E to get

$$E(t) = \frac{1}{k}\left(aM - De^{-kt}\right).$$

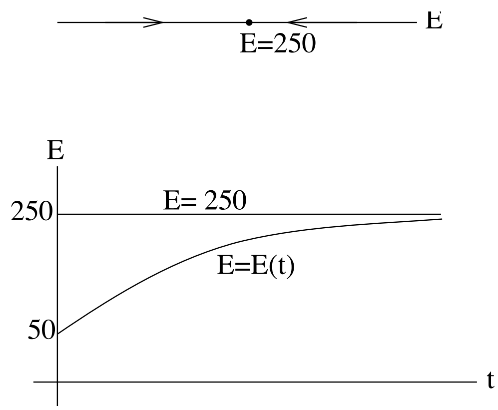

Figure 1: phase-line diagram and solution curve for question 5

The constant is given by the initial value as

$$E(0) = \frac{aM - D}{k}, \quad \text{or} \quad D = aM - kE(0) = 50 - 10 = 40.$$

Hence the solution is

$$E(t) = 5\left(50 - 40e^{-t/5}\right).$$

Solution version B

Steady state is $E^* = 160$. It is stable. In the long run, the concentration will approach 160 mg/l. Phase-line and solution curve similar to the above. Explicit solution is

$$E(t) = 4\left(40 - 30e^{-t/4}\right).$$

Solution version C

Steady state is $E^* = 150$. It is stable. In the long run, the concentration will approach 160 mg/l. Phase-line and solution curve similar to the above. Explicit solution is

$$E(t) = 3\left(50 - 40e^{-t/3}\right).$$

Question 6. [3 points] Solve the separable differential equation

$$\frac{dP}{dt} = \frac{1}{4+t^2}P, \qquad P(0) = 10.$$

Solution
Separating variables, we get

$$\frac{dP}{P} = \frac{1}{4+t^2}dt.$$

Integration on both sides yields

$$\ln|P| = \frac{1}{2}\arctan(t/2) + C.$$

Taking exponents on both sides, we get

$$P(t) = Ke^{\frac{1}{2}\arctan(t/2)}, \qquad K = e^C.$$

Using the initial value, we see that $\arctan(0) = 0$ and $P(0) = K$ so that the solution is

$$P(t) = 10e^{\frac{1}{2}\arctan(t/2)}.$$

Solution version B

$$P(t) = 5e^{\frac{1}{3}\arctan(t/3)}.$$

Solution version C

$$P(t) = 7e^{\frac{1}{4}\arctan(t/4)}.$$

Question 7. [6 points] Consider the autonomous differential equation

$$\frac{dP}{dt} = P(P^2 - 6P + 5).$$

Do not solve this equation explicitly!

1. (1 point) Find the steady states P_1^*, P_2^*, P_3^*.
2. (2 points) Find the stability of P_1^*, P_2^*, P_3^*, using the derivative test.
3. (1 point) Draw the phase line diagram.
4. (2 points) Without solving the equation explicitly, sketch the graph of $P(t)$ starting at $P(0) = 4$. Indicate inflections points if there are any.

Solution
We denote $f(P) = P(P^2 - 6P + 5)$. The zeros of f are the steady states of the differential equation, i.e.,

$$P_1^* = 0, P_2^* = 1, P_3^* = 5.$$

The derivative of f is

$$f'(P) = 3P^2 - 12P + 5. \qquad f'(0) = 5, f'(1) = -4, f'(5) = 20.$$

Hence P_1^* is unstable, P_2^* is stable and P_3^* is unstable. The phase-line diagram and solution curve are in Figure 2.

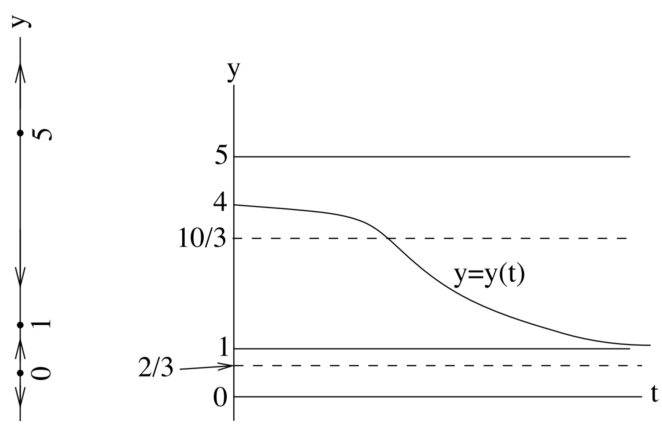

Figure 2: Phase-line and solution curve for question 7. Inflection point(s) on the dashed lines.

Solution version B
We denote $f(P) = P(P^2 - 9P + 20)$. The zeros of f are the steady states of the differential equation, i.e.,

$$P_1^* = 0, P_2^* = 4, P_3^* = 5.$$

The derivative of f is

$$f'(P) = 3P^2 - 18P + 20. \qquad f'(0) = 20, f'(4) = -4, f'(5) = 5.$$

Hence P_1^* is unstable, P_2^* is stable and P_3^* is unstable, see Figure 3.

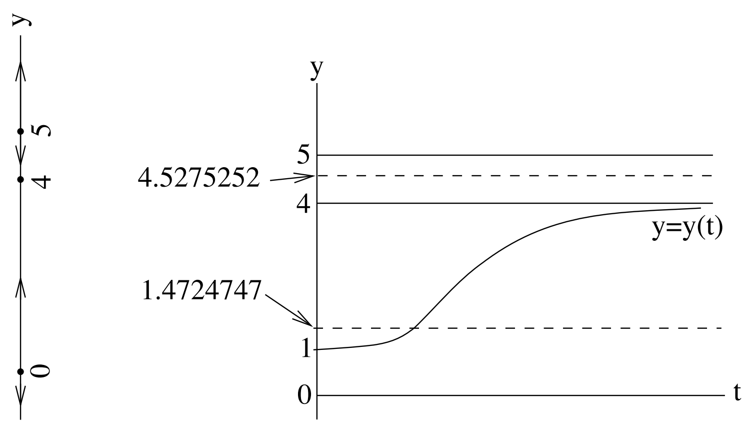

Figure 3: Phase-line and solution curve for question 7 version B. Inflection point(s) on the dashed lines.

Solution version C

We denote $f(P) = P(P^2 - 3P - 4)$. The zeros of f are the steady states of the differential equation, i.e.,

$$P_1^* = -1, P_2^* = 0, P_3^* = 4.$$

The derivative of f is

$$f'(P) = 3P^2 - 6P - 4. \qquad f'(-1) = 5, f'(0) = -4, f'(4) = 20.$$

Hence P_1^* is unstable, P_2^* is stable and P_3^* is unstable, see Figure 4.

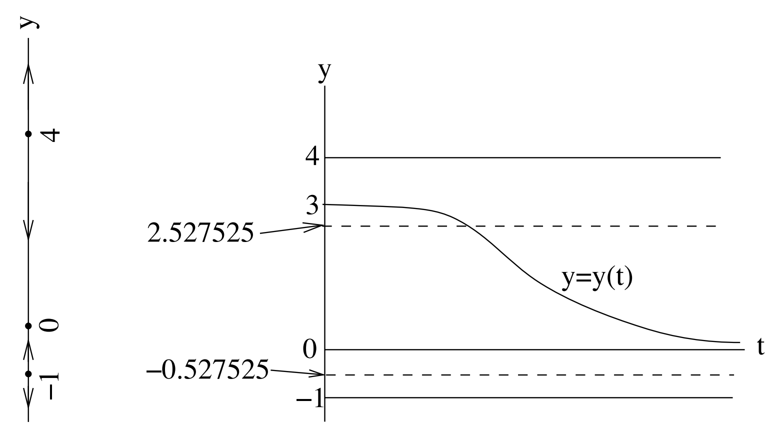

Figure 4: Phase-line and solution curve for question 7 version C. Inflection point(s) on the dashed lines.